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Best Quadrature Formulas and Splines

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1. INTRODUCTION

In this paper, best quadrature formulas in the sense of Sard with fixed knots corresponding to splines satisfying mixed boundary canditions are characterized. Work along these lines was initiated by Schoenberg in [7-9] and subsequently refined and generalized by Karlin in [3]. Here the analysis in [3] is extended to include quadrature formulas involving mixed boundary forms. Additionally, not only polynomial splines but also splines induced by a general differential operator of Polya type W are considered. This generality is useful because it reveals clearly, for the first time, the full role played by the adjoint differential operator in the correspondence between quadrature formulas and monosplines. Also, certain hypotheses made in [3] regarding sign consistency of matrices corresponding to the adjoint boundary forms are seen to be unnecessary, being consequences of the sign consistency already imposed on the original boundary forms (A result of this type is suggested by certain Green's function considerations.) This observation considerably simplifies the task of verifying that the hypothesis of some basic theorems in [3] are satisfied in concrete cases.

This paper is organized as follows. Section 2 contains the basic notation and concepts to be used. Section 3 establishes the basic correspondence between quadrature formulas exact on a specified class of polynomials and monosplines. Section 4 characterizes the quadrature formula best in the sense of Sard essentially in terms of an orthogonality condition (of little practical utility), and also by means of a system of linear equations explicitly available for computation. A simple example points out that Theorem 1.3 in [3] must be rephrased. Section 5 presents some important quadrature formulas involving mixed boundary forms. In particular, periodic and antiperiodic boundary forms are treated. Section 5 also contains a basic result (Theorem 5.2) concerning the sign consistency of boundary forms and their adjoints (Professor S. Karlin told me that he also discovered this result for the case of separated boundary conditions; his work is unpublished, but

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he lectured on it at the Weizmann Institute in 1973.) This result bears useful consequences here, and in the study of boundary value problems of the Sturm–Liouville type. Sections 6 and 7 deal with quadrature formulas involving separated boundary forms. In particular, improved versions of Theorems 3.2 and 4.1 in [3] are obtained Also, Theorem 6.3 extends and refines a basic result, Theorem 1 in Schoenberg [8]. Section 8 contains concluding remarks and extensions of this work, including a discussion of multiknot quadrature formulas.

2. Terminology and Preliminary Results

A *monospline of degree n* with knots $\{\xi_k\}_{k=1}^r$, $0 < \xi_1 < \cdots < \xi_r < 1$, is an expression of the form,

$$M(x) = \frac{x^n}{n!} + \sum_{\nu=1}^{n} b_\nu x^{\nu-1} + \sum_{k=1}^{r} d_k (x - \xi_k)_+^{n-1},$$

where b_ν and d_k are real. The class of such monosplines is denoted by $\mathcal{M}_{n,r}$. If the term $x^n/n!$ is discarded, the resulting function is a *spline of degree n* $-$ 1 with knots $\{\xi_k\}$. The linear space of these splines is denoted by $\mathcal{S}_{n,r}$. If the monosplines or splines are required to satisfy the boundary conditions $\mathcal{F}$, the resulting classes of functions will be denoted by $\mathcal{M}_{n,r}(\mathcal{F})$ and $\mathcal{S}_{n,r}(\mathcal{F})$, respectively. The knots $\{\xi_k\}$ remain fixed in what follows and so are not mentioned explicitly in the notation.

More precisely the splines (monosplines) above are polynomial splines (monosplines). They are piecewise solutions of the differential equation $Lu = 0$ where $L = D^n(L = D^{n+1})$, respectively. More generally (cf. Karlin and Studden [5]), consider splines and monosplines defined by means of the differential operator,

$$L = L_n = D_n D_{n-1} \cdots D_1,$$

where,

$$(D_j u)(x) = D[u(x)/w_j(x)], \qquad D = d/dx,$$

and,

$$w_j(x) > 0 \text{ on } 0 \leqslant x \leqslant 1,$$
$$w_j \in C^{2n+1-j}, \qquad j = 1, 2, \ldots, n.$$

Any solution to $Lu = 0$ is called an *L-polynomial* or *polynomial* for short.

The class of these polynomials is denoted by $\mathscr{P} = \mathscr{P}_n$. The differential equation $Lu = 0$ has a basis of solutions,

$$
\begin{aligned}
u_1(x) &= w_1(x), \\
u_2(x) &= w_1(x) \int_0^x w_2(t_1)\, dt_1, \\
&\cdots \\
u_n(x) &= w_1(x) \int_0^x w_2(t_1) \int_0^{t_1} w_3(t_2) \cdots \int_0^{t_{n-2}} w_n(t_{n-1})\, dt_{n-1} \cdots dt_1,
\end{aligned}
\tag{2.1}
$$

which constitute an extended complete Tchebycheff system, and satisfy the initial conditions,

$$(D^{j-1}u_i)(0) = w_i(0)\, \delta_{ij}, \qquad i, j = 1, \ldots, n.$$

where,

$$D^j = D_j D_{j-1} \cdots D_1 D_0, \qquad j = 1, \ldots, n,$$

and

$$D^0 = D_0 = I,$$

the identity operator. The function,

$$
\begin{aligned}
\phi_n(x; \xi) &= 0, & 0 \leq x < \xi \leq 1, \\
&= w_1(x) \int_\xi^x w_2(t_1) \int_\xi^{t_1} w_3(t_2) \cdots \int_\xi^{t_{n-2}} w_n(t_{n-1})\, dt_{n-1} \cdots dt_1, \\
& & 0 \leq \xi \leq x \leq 1,
\end{aligned}
\tag{2.2}
$$

is the fundamental solution for $Lu = 0$ determined by zero initial data at zero, and the characteristic jump discontinuity,

$$D^{n-1}\phi_n(\xi+; \xi) - D^{n-1}\phi_n(\xi-; \xi) = w_n(\xi),$$

which is equivalent to the requirement that the $(n-1)$st (ordinary) derivative of $\phi_n(x; \xi)$ exhibit a jump of $1/p_0(\xi)$ at $x = \xi$, where $L = p_0(x)\, d^n/dx^n + \cdots$.

An *L-spline*, or *spline* for short, with knots $\{\xi_k\}_{k=1}^r$ is a function $S \in C^{n-2}$ $[0, 1]$ which satisfies $(LS)(x) = 0$ except (possibly) when $x \in \{\xi_k\}$. Each such S has an explicit representation as,

$$S(x) = \sum_{\nu=1}^{n} b_\nu u_\nu(x) + \sum_{k=1}^{r} d_k \phi_n(x_j; \xi_k).$$

for certain constants $\{b_\nu\}$ and $\{d_k\}$. An *L-monospline*, or *monospline* for short, with knots $\{\xi_k\}_{k=1}^r$ is a function of the form,

$$M(x) = \psi_n(x) + \sum_{\nu=1}^{n} b_\nu u_\nu(x) + \sum_{k=1}^{r} d_k \phi_n(x_j; \xi_k),$$

where $\psi_n(x)$ is the unique solution to the initial value problem,

$$\begin{aligned} L_n \psi &= 1, \\ D^j \psi(0) &= 0, \qquad j = 0, 1,..., n-1. \end{aligned} \tag{2.3}$$

Direct integration yields the explicit representation for ψ_n ,

$$\psi_n(x) = w_1(x) \int_0^x w_2(t_1) \int_0^{t_1} w_3(t_2) \cdots \int_0^{t_{n-2}} w_n(t_{n-1}) \int_0^{t_{n-1}} dt_n \cdots dt_1 .$$

The differential operator L has adjoint,

$$L^* = L_n{}^* = D_1{}^* \cdots D_n{}^*,$$

where

$$D_j{}^* = (-1/w_j)\, D. \qquad j = 1,..., n.$$

Let

$$D^{*j} = D^*_{n+1-j} \cdots D_n{}^*, \qquad j = 1,..., n.$$

and by special convention

$$D^{*0} = D^*_{n+1} D_n{}^* = I.$$

For later purposes it is useful to introduce the following notation:

$$D_j = D(1/w_{2n+2-j}) \qquad j = n+1,..., 2n,$$

where,

$$w_{n+1}(x) = 1, \qquad 0 \leqslant x \leqslant 1.$$

Then,

$$L^* = ((-1)^n/w_1)\, D_{2n} \cdots D_{n+1} ,$$

and,

$$L^*L = ((-1)^n/w_1)\, D_{2n} \cdots D_1 .$$

Thus $L^*Lu = 0$ has a basis,

$$u_i(x) = w_1(x) \int_0^x w_2(t_1) \int_0^{t_1} w_3(t_2) \cdots \int_0^{t_{i-2}} w_i(t_{i-1})\, dt_{i-1} \cdots dt_1 \qquad (2.4)$$

where $i = 1,..., 2n$. (Notice that $u_1,..., u_n$ are as given in (2.1).) Also, $L^*u = 0$ has a basis constructed as in (2.1) with $w_1,..., w_n$ replaced, respectively, by $w_{n+1},..., w_{2n}$. This basis is denoted by,

$$u_1^*,..., u_n^*,$$

and the fundamental solution for L^* is denoted by,

$$\phi_n^*(x; \xi).$$

It is determined by zero initial data at zero and the characteristic jump continuity,

$$D^{*n-1}\phi_n^*(\xi+; \xi) - D^{*n-1}\phi_n^*(\xi-; \xi) = -w_1(\xi).$$

Just as for L, polynomials, splines, and monosplines are induced by L^* and L^*L. These L^*-splines and L^*-monosplines play a key role in determining quadrature formulas exact for L-polynomials, as do the L^*L-splines and L^*L-monosplines. In particular, an L^*-monospline has the form,

$$M(x) = \psi_n^*(x) + \sum_{\nu=1}^{n} b_\nu u_\nu^*(x) + \sum_{k=1}^{r} d_k \phi_n^*(x; \xi_k),$$

where ψ_n^* is the unique solution to,

$$L_n^* \psi = 1,$$
$$D^{*j}\psi(0) = 0, \qquad j = 0,..., n-1,$$

and an $L*L$-monospline has the form

$$N(x) = \psi_{2n}(x) + \sum_{\nu=1}^{2n} b_\nu u_\nu(x) + \sum_{k=1}^{r} d_k \phi_{2n}(x; \xi_k),$$

where ψ_{2n} and ϕ_{2n} are defined in terms of $w_1,..., w_{2n}$ just as ψ_n and ϕ_n are defined in terms of $w_1,..., w_n$.

In order to simplify notation the following conventions will be used. The class of L-polynomials, L-splines, and L-monosplines will always be denoted, respectively, by

$$\mathscr{P}_n, \mathscr{S}_{n,r}, \text{ and } \mathscr{M}_{n,r}.$$

The class of all L^*-polynomials, L^*-splines, and L^*-monosplines will be denoted, respectively, by

$$\mathscr{P}_n^*, \mathscr{S}_{n,r}^*, \mathscr{M}_{n,r}^*$$

The class of all L^*L-polynomials, L^*L-splines, and L^*L-monosplines will be denoted, respectively, by

$$\mathscr{P}_{2n}, \mathscr{S}_{2n,r}, \mathscr{M}_{2n,r}.$$

If any of these classes are subject to boundary constraints, $\mathscr{F}$, notation such as $\mathscr{S}_{n,r}(\mathscr{F})$ will indicate this fact.

Consider mixed boundary forms,

$$U_i(u) = \sum_{j=1}^{n} a_{ij} D^{j-1} u(0) + \sum_{j=1}^{n} b_{ij} D^{j-1} u(1), \qquad i = 1,\ldots, p.$$

Let

$$U = (U_1, \ldots, U_p)$$

be the vector of these boundary forms and

$$C = [A, B],$$

be the $p \times 2n$ matrix determined by these forms. Assume rank $C = p$.

Let $v \in C^{n-2}[0, 1]$ with $v^{(n-1)}$ piecewise continuous with at worst jump discontinuities at $\xi_1, \ldots, \xi_r$. Then for $u \in C^{n-1}[0, 1]$ with $u^{(n)}$ piecewise continuous with at worst jump discontinuities integration by parts yields

$$\int_0^1 (Lu)\, v\, dx = B(u, v) + \sum_{k=1}^{r} \frac{[D^{*^{n-1}} v(\xi_k^-) - D^{*^{n-1}} v(\xi_k^+)]}{w_1(\xi_k)} u(\xi_k) + \int_0^1 u(L^* v)\, dx$$

where

$$B(u, v) = \sum_{j=0}^{n-1} \frac{(D^{n-1-j} u)(x)(D^{*j} v)(x)}{w_{n-j}(x)} \Big|_0^1.$$

For $z \in C^{n-1}$ in a neighborhood 0 and 1 in [0, 1] define the $2n$-vectors $\bar{z}$ and z^* by

$$\bar{z} = (D^0 z(0), \ldots, D^{n-1} z(0), D^0 z(1), \ldots, D^{n-1} z(1))^T,$$

$$z^* = (D^{*0} z(0), \ldots, D^{*^{n-1}} z(0), D^{*0}(1), \ldots, D^{*^{n-1}} z(1))^T.$$

Define matrices $B(x)$ and S

$$B(x) = \left\| \frac{1}{w_1(x)}\,\delta_n, \frac{1}{w_2(x)}\,\delta_{n-1}, \ldots, \frac{1}{w_n(x)}\,\delta_1 \right\|_{n\times n},$$

where $\delta_i = (0,\ldots, 0, 1, 0,\ldots, 0)^T$ is the usual ith coordinate basis vector, and

$$S = \left\| \begin{matrix} -B(0) & 0 \\ 0 & B(1) \end{matrix} \right\|_{2n\times 2n}.$$

Then

$$B(u, v) = S\bar{u} \cdot v^*,$$

where $\cdot$ is the usual inner product in $2n$-space.

Adjoin rows $p+1,\ldots, 2n$ to the matrix $C = \| A, B \|$ so that the resulting matrix

$$\hat{C} = \| \hat{A}, \hat{B} \|_{2n\times 2n}$$

has rank $2n$. Boundary forms complementary to U are defined by

$$\hat{C}\bar{u} = \left\| \begin{matrix} U(u) \\ U_c(u) \end{matrix} \right\| \begin{matrix} \}\,p \\ \}\,2n-p \end{matrix}.$$

Forms $U_c^*(v)$ and $U^*(v)$ are defined by

$$\begin{matrix} p\{ \\ 2n-p\{ \end{matrix} \left\| \begin{matrix} U_c^*(v) \\ U^*(v) \end{matrix} \right\| = (\hat{C}^{-1})^* S^* v^*. \tag{2.5}$$

where the $*$ on the matrices signifies the transpose conjugate operation. The forms $U^*(v)$ are *adjoint* to the forms $U(u)$ because

$$B(u, v) = S\bar{u} \cdot v^* = \begin{bmatrix} U(u) \\ U_c(u) \end{bmatrix} \cdot \begin{bmatrix} U_c^*(v) \\ U^*(v) \end{bmatrix},$$

and consequently

$$\begin{aligned} \int_0^1 (Lu)\, v\, dx &= U(u) \cdot U_c^*(v) + U_c(u) \cdot U^*(v) \\ &\quad + \sum_{k=1}^{r} [D^{*^{n-1}} v(\xi_k^+) - D^{*^{n-1}} v(\xi_k^-)]\, u(\xi_k)/w_1(\xi_k) + \int_0^1 u(L^* v)\, dx. \end{aligned} \tag{2.6}$$

This equation will be called the *basic integration by parts formula*.

Let A be an $m \times n$ matrix and $p \leqslant m, n$. Then,

$$A\begin{pmatrix} i_1, \dots, i_p \\ j_1, \dots, j_p \end{pmatrix},$$

stands for the determinant of the matrix obtained from A by deleting all rows and columns except those labelled $i_1, \dots, i_p$ and $j_1, \dots, j_p$, respectively.

3. Quadrature Formulas Exact on L-Polynomials

The correspondence between monosplines and quadrature formulas exact on (ordinary) polynomials of degree $\leqslant n - 1$ introduced in Schoenberg [8] and extended and refined in Karlin [3] to embrace general separated boundary forms will be extended to include general mixed boundary forms. Also, the analysis here is presented for quadrature formulas exact on L-polynomials. In this setting, it emerges clearly, for the first time, that quadrature formulas exact on L-polynomials are in 1 : 1 correspondence with certain L^*-monosplines. In the ordinary polynomial case $L^* = (-1)^n L$, the set of monosplines for L and L^* agree or differ by a minus sign and the full role played by adjoint differential operator is obscured. The analysis of this section differs in several respects from that in [3, 7] because certain direct evaluations possible in the ordinary polynomial case are not available.

Let

$$M(x) = \psi_n^*(x) + \sum_{\nu=1}^{n} b_\nu u_\nu^*(x) + \sum_{k=1}^{r} d_k \phi_n^*(x; \xi_k),$$

be an L^*-monospline. Replacing v by M in (2.6) and utilizing $L^*M(x) = 1$ for $x \notin \{\xi_k\}$ it follows that

$$\begin{aligned} \int_0^1 u\,dx = & -U(u) \cdot U_c^*(M) - U_c(u) \cdot U^*(M) \\ & + \sum_{k=1}^{r} [D^{*^{n-1}} M(\xi+) - D^{*^{n-1}} M(\xi-)]\, u(\xi_k)/w_1(\xi_k) + \int_0^1 (Lu)\, M\,dx. \end{aligned} \tag{3.1}$$

If M also satisfies the adjoint boundary conditions,

$$U^*(M) = 0$$

then

$$\int_0^1 u\,dx = \sum_{i=1}^{p} a_i U_i(u) + \sum_{k=1}^{r} c_k u(\xi_k) + \int_0^1 (Lu)\, M\,dx \tag{3.2}$$

where

$$\begin{aligned} a_i &= U^*_{c,i}(M), & i &= 1,\dots, p, \\ c_k &= \frac{D^{*^{n-1}}M(\xi_k^-) - D^{*^{n-1}}M(\xi_k^+)}{w_1(\xi_k)}, & k &= 1,\dots, r. \end{aligned} \tag{3.3}$$

Consequently the quadrature formula

$$Q(u) = \sum_{i=1}^{p} a_i U_i(u) + \sum_{k=1}^{r} c_k u(\xi_k),$$

is exact on L-polynomials.

Conversely suppose

$$Q(u) = \sum_{i=1}^{p} a_i' U_i(u) + \sum_{k=1}^{r} c_k' u(\xi_k) \tag{3.4}$$

is exact on L-polynomials. For $u \in C^n[0, 1]$ the (generalized) Taylor formula

$$u(x) = \sum_{i=1}^{n} e_i u_i(x) + \int_0^1 Lu(t)\, \phi_n(x; t)\, dt,$$

$$e_i = \frac{D^{i-1}u(0)}{w_i(0)}, \qquad i = 1,\dots, n,$$

holds. (Consult [5, Chapter 11, Lemma 2.2] and its proof.) Hence if

$$R(u) = \int_0^1 u\, dx - Q(u)$$

is the error functional for the quadrature formula, it follows that

$$R(u) = \int_0^1 Lu(t)\, R_x\phi_n(x; t)\, dt,$$

where the subscript indicates that R operatores with respect to the variable x, and the interchange of order is easily justified. The next two lemmas show that

$$M(t) = R_x\phi_n(x; t)$$

is an L^*-monospline. Observe that

$$R_x\phi_n(x; t) = \int_0^1 \phi_n(x; t)\, dx - Q_x[\phi_n(x; t)].$$

LEMMA 3.1. $L^*[\int_0^1 \phi_n(x; t)\, dx] = 1$; *hence,* $\int_0^1 \phi_n(x; t)\, dx$ *differs from the* *unique solution* ψ_n^* *of* $L^*\psi = 1$, $D^{*j}\psi(0) = 0$, $j = 0,\ldots, n-1$, *by an* L^*-*polynomial.*

Proof. A simple calculation yields

$$(d/dt)\, \phi_n(x; t) = -w_n(t)\, \phi_{n-1}(x; t),$$

where $\phi_{n-1}(x; t)$ is the fundamental solution corresponding to $D_{n-1} \cdots D_1$. Since $L^* = D_1^* \cdots D_n^*$ where $D_j^* = (-1/w_j)\, D$, it follows that

$$D_n^* \int_0^1 \phi_n(x; t)\, dx = \frac{-1}{w_n(t)} \frac{d}{dt} \int_t^1 \phi_n(x; t)\, dx$$

$$= \int_t^1 \phi_{n-1}(x; t)\, dx.$$

Repeated differentiation yields

$$D_2^* \cdots D_n^* \int_0^1 \phi_n(x; t)\, dt = \int_t^1 w_1(x)\, dx$$

and

$$L^* \int_0^1 \phi_n(x; t)\, dx = 1.$$

The final assertion in the lemma is evident. ∎

LEMMA 3.2. $Q_x[\phi_n(x; t)]$ *is an* L^*-*spline.*

Proof. First,

$$Q_x[\phi_n(x; t)] = \sum_{i=1}^{n} a_i' U_{i,x}[\phi_n(x; t)] + \sum_{k=1}^{r} c_k' \phi_n(\xi_k : t).$$

Consider a typical term in the first sum.

$$U_{i,x}[\phi_n(x; t)] = \sum_{j=1}^{n} b_{ij} D_x^{j-1} \phi_n(x; t) \Big|_{x=1}. \tag{3.5}$$

for $0 < t < 1$. A short calculation yields,

$$h_j(t) = D_x^{j-1} \phi_n(x; t) \big|_{x=1} = w_j(1) \int_t^1 w_{j+1}(t_j) \int_t^{t_j} \cdots \int_t^{t_{n-2}} w_n(t_{n-1})\, dt_{n-1} \cdots dt_j.$$

Evidently,

$$D_n^* h_j(t) = w_j(1) \int_t^1 w_{j+1}(t_j) \int_t^{t_j} \cdots \int_t^{t_{n-3}} w_{n-1}(t_{n-2})\, dt_{n-2} \cdots dt_j,$$

and so upon successive application of $D_{n-1}^*, \ldots, D_1^*$ it follows that

$$L^* h_j(t) = 0$$

for $j = 1, \ldots, n$. Thus the first sum in $Q_x[\phi_n(x; t)]$ is an L^*-polynomial. (When $t = 0$ the right side of (3.5) should be increased by $\sum_{i=1}^{p} a_i[a_{in} w_n(0)]$ which L^* annihilates.)

Consider a typical term in the second sum in $Q_x[\phi_n(x; t)]$,

$$\begin{aligned} \phi_n(\xi_k; t) &= w_1(\xi_k) \int_t^{\xi_k} w_2(t_1) \int_t^{t_1} w_3(t_2) \cdots \int_t^{t_{n-2}} w_n(t_{n-1})\, dt_{n-1} \cdots dt_1, \qquad t \leq \xi_k, \\ &= 0, \qquad \xi_k < t. \end{aligned}$$

Differentiating as in Lemma 3.1 yields

$$\begin{aligned} D_2^* \cdots D_n^* \phi_n(\xi_k; t) &= w_1(\xi_k), \qquad t \leq \xi_k, \\ &= 0, \qquad \xi_k < t. \end{aligned}$$

and $L^* \phi_n(\xi_k; t) = 0$ for $t \neq \xi_k$. Thus,

$$D^{*^{n-1}} \phi_n(\xi_k; \xi_k^+) - D^{*^{n-1}} \phi_n(\xi_k; \xi_k^-) = -w_1(\xi_k),$$

and $\phi_n(\xi_k; t)$ exhibits the same jump in its $(n-1)$st derivative as the fundamental solution $\phi_n^*(t; \xi_k)$. Consequently, $\phi_n^*(t; \xi_k)$ and $\phi_n(\xi_k; t)$ differ by an L^*-polynomial. These observations prove the lemma. ∎

Lemmas 3.1 and 3.2 establish that the remainder functional for the quadrature formula (3.4) can be expressed as

$$R(u) = \int_0^1 (Lu) M \, dx \tag{3.6}$$

for some monospline $M \in \mathscr{M}_{n,r}^*$. On the other hand, from (3.1)

$$\int_0^1 u \, dx = \sum_{i=1}^{p} a_i U_i(u) + \sum_{i=p+1}^{2n} b_i U_{c,i}(u) + \sum_{k=1}^{r} c_k u(\xi_k) + \int_0^1 (Lu) M \, dx \tag{3.7}$$

where a_i, c_k are given by (3.3) and

$$b_i = -U_i^*(M), \qquad i = p+1, \ldots, 2n. \tag{3.8}$$

Now (3.4), (3.6)–(3.8) yield

$$\sum_{i=1}^{p} (a_i - a_i') U_i(u) + \sum_{i=p+1}^{2n} b_i U_{c,i}(u) + \sum_{k=1}^{r} (c_k - c_k') u(\xi_k) = 0 \tag{3.9}$$

for all $u \in C^n[0, 1]$.

The $2n$ boundary conditions $U(u) = 0$, $U_c(u) = 0$ are equivalent to the stipulations

$$D^{j-1}u(x) = 0, \qquad j = 1,..., n; x = 0, 1.$$

For all $u \in C^n[0, 1]$ satisfying these requirements, (3.9) reduces to

$$\sum_{k=1}^{r} (c_k - c_k')\, u(\xi_k) = 0$$

which manifestly implies

$$c_k' = c_k \qquad k = 1,..., r.$$

Thus (3.9) reduces to

$$\sum_{i=1}^{p} (a_i - a_i')\, U_i(u) + \sum_{i=p+1}^{2n} b_i U_{c,i}(u) = 0 \tag{3.10}$$

for all $u \in C^n[0, 1]$. In particular, for $u = u_i$ with u_i given in (2.4),

$$\sum_{j=1}^{p} U_j(u_i)(a_j - a_j') + \sum_{j=p+1}^{2n} U_{c,j}(u_i)\, b_j = 0 \tag{3.11}$$

for $i = 1,..., 2n$. This $2n \times 2n$ system has matrix

$$\left\| \begin{matrix} U_j(u_i) & U_{c,j}(u_i) \\ i = 1,..., 2n & i = 1,..., 2n \\ j = 1,..., p & j = p + 1,..., 2n \end{matrix} \right\|,$$

which is the transpose of the matrix

$$\hat{C} \| \bar{u}_1 ,..., \bar{u}_{2n} \|$$

where $\hat{C}$ and $\bar{u}_i$ are defined above (2.5) in Section 2, and $\bar{u}_i$ is the ith column of the indicated matrix. Since $\hat{C}$ is nonsingular the coefficient matrix of (3.11) will be nonsingular provided

$$\det \| \bar{u}_1 ,..., \bar{u}_{2n} \| \neq 0.$$

Now

$$\| \bar{u}_1 ,..., \bar{u}_{2n} \| = \left\| \begin{matrix} W(u_1 ,..., u_n)(0) & W(u_{n+1} ,..., u_{2n})(0) \\ W(u_1 ,..., u_n)(1) & W(u_{n+1} ,..., u_{2n})(1) \end{matrix} \right\|$$

where

$$W(z_1 ,..., z_n)(x) = \|(D^{i-1}z_j)(x)\|_{i,j=1,...,n}.$$

From (2.1) and (2.4)

$$W(u_1, \dots, u_n)(0) = \mathrm{diag}(w_1(0), \dots, w_n(0)),$$
$$W(u_{n+1}, \dots, u_{2n})(0) = 0.$$

Hence, the system (3.11) is nonsingular iff

$$\det W(u_{n+1}, \dots, u_{2n})(1) \neq 0$$

which is the case because the kernel $\Phi(i, x) = u_i(x)$, $i = 1, \dots, 2n$ and $0 < x \leqslant 1$, is $\mathrm{ETP}_n(x)$ (see [1, Chapter 6, Theorem 1.2]) and so

$$\det W(u_{n+1}, \dots, u_{2n})(1) > 0.$$

Consequently, (3.11) implies that

$$a_i' = a_i, \qquad i = 1, \dots, p,$$
$$b_i = 0, \qquad i = p+1, \dots, 2n.$$

Thus, the quadrature formula (3.4) is induced by a monospline $M \in \mathscr{M}^*_{n,r}$ which satisfies the adjoint boundary conditions $U^*(M) = 0$.

If $M_1, M_2 \in \mathscr{M}^*_{n,r}$ both generate the same quadrature formula, then

$$\int_0^1 (Lu)(M_1 - M_2)\, dx = 0$$

for all $u \in C^n[0, 1]$. Since $LC^n[0, 1] = C[0, 1]$, it follows that $M_1 = M_2$. The following theorem has been established.

THEOREM 3.1. *There is a 1 : 1 correspondence between quadrature formulas of the form*

$$Q(u) = \sum_{i=1}^{p} a_i U_i(u) + \sum_{k=1}^{r} c_k u(\xi_k) \tag{3.12}$$

which are exact on L-polynomials and L^*-monosplines, M, satisfying the adjoint boundary conditions, $U^*(M) = 0$. If $Q(u)$ corresponds to M, then

$$a_i = -U^*_{c,i}(M), \qquad i = 1, \dots, p,$$

$$c_k = \frac{D^{*n-1}M(\xi_k^+) - D^{*n-1}M(\xi_k^-)}{w_1(\xi_k)}, \qquad k = 1, \dots, r,$$

$$R(u) = \int_0^1 (Lu)\, M\, dx.$$

4. Best Quadrature Formulas

Let $\mathscr{C}$ be the class of quadrature formulas (3.12) which are exact on L-polynomials. Let $\mathscr{B}$ [resp., $\mathscr{B}^*$] be the class of functions satisfying the boundary conditions $U(u) = 0$ [resp., $U^*(u) = 0$].

A quadrature formula $\tilde{Q} \in \mathscr{C}$ is *best in the sense of Sard* for the class $\mathscr{C}$ if

$$\sup_{\|u\|_L \leqslant 1} |\tilde{R}(u)| = \inf_{Q \in \mathscr{C}} \sup_{\|u\|_L \leqslant 1} |R(u)|, \tag{4.1}$$

where

$$\|u\|_L^2 = \int_0^1 [Lu(x)]^2 \, dx$$

and $\tilde{R}(u)$, $R(u)$ are the respective error functionals for the quadrature formulas Q, $\tilde{Q}$.

The analysis leading up to Theorem 4.1 below is due to Karlin in [3]. (Theorem 4.1 below is essentially Theorem 1.2 in [3].) In view of the error formula

$$R(u) = \int_0^1 (Lu)\, M \, dx,$$

the Schwarz inequality together with the condition for equality, and the fact that L maps $C^n[0, 1]$ onto $C[0, 1]$, (4.1) is equivalent to

$$\int_0^1 |\tilde{M}(x)|^2 \, dx = \min_{\mathscr{M}_{n,r}^*(\mathscr{B}^*)} \int_0^1 |M(x)|^2 \, dx \tag{4.2}$$

where $\tilde{M} \in \mathscr{M}_{n,r}^*(\mathscr{B}^*)$ corresponds to $\tilde{Q}$. Thus the problem of finding a quadrature formula best in the sense of Sard is equivalent to finding a monospline in $\mathscr{M}_{n,r}^*(\mathscr{B}^*)$ which best approximates zero in $L_2[0, 1]$. Since $\mathscr{M}_{n,r}^*(\mathscr{B}^*)$ is closed and convex, this problem has a unique solution $\tilde{M}$ provided $\mathscr{M}_{n,r}^*(\mathscr{B}^*)$ is nonempty (equivalently, $\mathscr{C}$ is nonempty). Furthermore, since $\mathscr{M}_{n,r}^*(\mathscr{B}^*)$ is the translate of the subspace $\mathscr{S}_{n,r}^*(\mathscr{B}^*)$ by $\tilde{M}$, (4.2) states that 0 is the best approximation to $\tilde{M}$ in $\mathscr{S}_{n,r}^*(\mathscr{B}^*)$. Thus $\tilde{M}$ is characterized by the orthogonality requirement,

$$\int_0^1 \tilde{M}(x)\, S(x) \, dx = 0, \qquad S \in \mathscr{S}_{n,r}^*(\mathscr{B}^*).$$

The following theorem has been proved.

THEOREM 4.1. *Assume $\mathscr{C}$ is not empty (see Theorem 4.2). Then the* *quadrature formula best in the sense of Sard corresponds to the unique monospline* $\tilde{M} \in \mathscr{M}_{n,r}^*(\mathscr{B}^*)$ *determined by the orthogonality condition*

$$\int_0^1 \tilde{M}(x)\, S(x)\, dx = 0, \qquad S \in \mathscr{S}_{n,r}^*(\mathscr{B}^*).$$

The orthogonality condition in Theorem 4.1 does not provide a practical characterization of $\tilde{M}$ because a basis for $\mathscr{S}_{n,r}^*(\mathscr{B}^*)$ is not readily at hand in most cases. The following result, Theorem 4.2, provides a useful practical determination of $\tilde{M}$ in terms of an explicitly available system of linear equations. Theorem 4.2 is the extension of Theorem 3.1 in [3] to the case of mixed boundary forms.

For $v = S \in \mathscr{S}_{n,r}^*(\mathscr{B}^*)$ the basic integration by parts formula (2.6) yields

$$\int_0^1 (Lu)\, S\, dx = U(u) \cdot U_c^*(S) + \sum_{k=1}^{r} \frac{D^{*^{n-1}}S(\xi_k^-) - D^{*^{n-1}}S(\xi_k^+)}{w_1(\xi_k)}\, u(\xi_k) \tag{4.3}$$

because $L^*S(x) = 0$ for $x \notin \{\xi_k\}$.

THEOREM 4.2. *The class* $\mathscr{C}$ *of admissible quadrature formulas is nonempty if there exists a monospline* $\tilde{N} \in \mathscr{M}_{2n,r}$ *(see Section 2 for the notation) such that*

$$\begin{aligned} U(\tilde{N}) &= 0, \\ U^*(L\tilde{N}) &= 0, \\ \tilde{N}(\xi_k) &= 0, \qquad k = 1,\ldots, r, \end{aligned} \tag{4.4}$$

in which case

$$\tilde{M} = L\tilde{N}$$

determines the quadrature formula best in the sense of Sard.

Proof. If $\tilde{N}$ satisfies (4.4), then $\tilde{M} = L\tilde{N} \in \mathscr{M}_{n,r}^*(\mathscr{B}^*)$. (Indeed it is easy to confirm that $L\mathscr{M}_{2n,r} \in \mathscr{M}_{n,r}^*$). Hence the class $\mathscr{C}$ is nonempty. From (4.3) with $u = \tilde{N}$

$$\int_0^1 \tilde{M} S\, dx = 0, \qquad S \in \mathscr{S}_{n,r}^*(\mathscr{B}^*) \tag{4.5}$$

and $\tilde{M}$ determines the best quadrature formula. ∎

Let $\tilde{N} = \psi_{2n} + \tilde{S}$ where $\tilde{S} \in \mathscr{S}_{2n,r}$. Then (4.4) is equivalent to

$$\begin{aligned} U(\tilde{S}) &= -U(\psi_{2n}), \\ U^*(L\tilde{S}) &= -U^*(L\psi_{2n}), \\ \tilde{S}(\xi_k) &= -\psi_{2n}(\xi_k), \qquad k = 1,\ldots, r. \end{aligned} \tag{4.6}$$

Theorem 1.3 in [3] states that $\mathscr{C}$ is nonempty iff the determinant of the system (4.6) is nonzero. This result must be rephrased in view of the following example; in fact, the reasoning used in Theorem 1.3 in [3] is essentially that used to prove Theorem 4.3 below. Consider quadrature formulas of the form

$$Q_c(u) = cu(\tfrac{1}{2}).$$

Among these quadrature formulas precisely one, $Q_1(u)$, is exact on polynomials of degree $\leqslant n - 1$ where $n = 2$. (Here the ordinary polynomial case is treated with $L = d^2/dx^2$.) Of course, $Q_1(u)$ is just the familiar midpoint rule. In this case conditions (4.4) on $\tilde{N} \in \mathscr{M}_{4,1}$ are,

$$U^*(\tilde{N}'') = 0,$$
$$\tilde{N}(\tfrac{1}{2}) = 0,$$

where $U^*(u) = 0$ is: $u(0) = u'(0) = u(1) = u'(1) = 0$. A short computation yields

$$\tilde{N}(x) = \frac{x^4}{4!} - \frac{2^{-4}}{4!} + B(x - 1/2) - \frac{1}{6}(x - 1/2)_+^3$$

with B an arbitrary constant. Consequently, (4.6) must have a zero determinant. Finally,

$$\tilde{M}(x) = \tilde{N}''(x) = (x^2/2!) - (x - \tfrac{1}{2})_+$$

and it is easily checked that $\tilde{M}$ determines $Q_1(u)$ as it must.

In view of this example it is useful to determine when (4.4), equivalently (4.6), determines $\tilde{N}$ uniquely.

THEOREM 4.3. *The requirements* (4.4), equivalently (4.6), *determine* $\tilde{N}$ *uniquely iff the only polynomial in* $\mathscr{P}_n(\mathscr{B})$ *interpolating zero data on* $\{\xi_k\}_{k=1}^r$ *is the zero polynomial. Thus,* $\tilde{N}$ *is unique when* $r \geqslant n$.

Proof. Suppose (4.4) uniquely determines $\tilde{N} \in \mathscr{M}_{2n,r}$. If $P \in \mathscr{P}_n(\mathscr{B})$ interpolates zero data on $\{\xi_k\}$, then $N_1 = \tilde{N} + P$ satisfies (4.4). Hence $\tilde{N} = N_1$ and $P = 0$.

Conversely assume zero is the only polynomial in $\mathscr{P}_n(\mathscr{B})$ interpolating zero data on $\{\xi_k\}$. Let $S_0 \in \mathscr{S}_{2n,r}$ be a solution to the homogeneous system

$$U(S_0) = 0,$$
$$U^*(LS_0) = 0,$$
$$S_0(\xi_k) = 0, \quad k = 1,..., r,$$

corresponding to (4.6) and let $S_1 = LS_0$. Clearly, $S_1 \in \mathscr{S}^*_{n,r}(\mathscr{B}^*)$. From (4.3)

$$\int_0^1 S_1(x)\, S(x)\, dx = 0, \qquad S \in \mathscr{S}^*_{n,r}(\mathscr{B}^*).$$

Thus $S_1 = 0$, i.e., $LS_0 = 0$ and so $S_0 \in \mathscr{S}_n(\mathscr{B})$ interpolates zero data on $\{\xi_k\}$. Hence, $S_0 = 0$ and (4.6) has a unique solution. ▮

Remark 4.1. Conditions (4.4) in Theorem 4.2 are sufficient to ensure that the class of admissible quadrature formulas, $\mathscr{C}$, is nonempty; however, it is not known whether these conditions are necessary as well. The following conditions, rather close to (4.4), are both necessary and sufficient for $\mathscr{C}$ to be nonempty,

$$\begin{gathered} U(\tilde{N}) = 0, \\ U^*(L\tilde{N}) = 0, \\ \sum_{k=1}^{r} \frac{[D^{*n-1}S(\xi_k^-) - D^{*n-1}S(\xi_k^+)]}{w_1(\xi_k)}\, \tilde{N}(\xi_k) = 0, \qquad S \in \mathscr{S}^*_{n,r}(\mathscr{B}^*). \end{gathered} \tag{4.7}$$

If all these conditions are satisfied $\tilde{M} = L\tilde{N}$ corresponds to the best quadrature formula as is seen by the argument of Theorem 4.2. On the other hand, if $\mathscr{C}$ is nonempty and $\tilde{M} \in \mathscr{S}^*_{n,r}(\mathscr{B}^*)$ determines the best quadrature formula, then the boundary value problem,

$$\begin{aligned} L\tilde{N} &= \tilde{M}, \\ U(\tilde{N}) &= 0, \end{aligned}$$

is solvable because the orthogonality property (4.5) guarantees that $\tilde{M}$ is orthogonal to all solutions of the homogeneous adjoint boundary value problem

$$\begin{aligned} L^*v &= 0, \\ U^*(v) &= 0. \end{aligned}$$

It is easy to check that $\tilde{N} \in \mathscr{M}_{2n,r}$ and also $U^*(L\tilde{N}) = U^*(\tilde{M}) = 0$. Finally from (4.3), (4.5)

$$0 = \int_0^1 \tilde{M}S\, dx = \sum_{k=1}^{r} \frac{[D^{*n-1}S(\xi_k^-) - D^{*n-1}S(\xi_k^+)]}{w_1(\xi_k)}\, \tilde{N}(\xi_k)$$

for $S \in \mathscr{S}^*_{n,r}(\mathscr{B}^*)$, and conditions (4.7) hold.

In the next section, conditions (4.4) are shown to uniquely determine $\tilde{N}$ for some important classes of quadrature formulas involving mixed boundary forms.

5. Some Important Quadrature Formulas with Mixed Boundary Forms

The determination of $\tilde{M}$ corresponding to the best quadrature formula from (4.4) involves solving the $(2n + r) \times (2n + r)$ system (4.6). In expanded form this system is

$$\sum_{j=1}^{n} a_{ij} D^{j-1} S(0) + \sum_{j=1}^{n} b_{ij} D^{j-1} S(1) = e_i, \qquad i = 1,..., p,$$

$$\sum_{j=1}^{n} a_{ij}^* D^{*j-1} LS(0) + \sum_{j=1}^{n} b_{ij}^* D^{*j-1} LS(1) = f_i, \qquad i = p + 1,..., 2n, \quad (5.1)$$

$$S(\xi_k) = 0, \qquad k = 1,..., r,$$

where $S(x) = \sum_{i=1}^{2n} a_i u_i(x) + \sum_{k=1}^{r} d_k \phi_{2n}(x; \xi_k) \in \mathscr{S}_{2n,r}$ and

$$e = (e_i) = -U(\psi_{2n}),$$
$$f = (f_i) = -U^*(L\psi_{2n}).$$

Here

$$A_* = \| a_{ij}^* \|, \qquad B_* = \| b_{ij}^* \|$$

are the matrices such that $\| A_*, B_* \|$ is the matrix of the adjoint boundary forms $U^*(u)$ constructed in (2.5). From Section 2,

$$D^{*j-1} = \frac{(-1)^{j-1}}{w_{n+2-j}} D_{n+j-1} \cdots D_{n+1}$$

and the boundary conditions in (5.1) can be expressed in the more convenient form

$$\sum_{j=1}^{n} a_{ij} D^{j-1} S(0) + \sum_{j=1}^{n} b_{ij} D^{j-1} S(1) = e_i,$$
$$i = 1,..., p,$$

$$\sum_{j=1}^{n} \frac{a_{ij}^*(-1)^{j-1}}{w_{n+2-j}(0)} D^{n+j-1} S(0) + \sum_{j=1}^{n} \frac{b_{ij}^*(-1)^{j-1}}{w_{n+2-j}(1)} D^{n+j-1} S(1) = f_i,$$
$$i = p + 1,..., 2n. \quad (5.2)$$

To guarantee the existence of a spline $S \in \mathscr{S}_{2n,r}$ satisfying (5.1) appeal is made to the basic interpolation theorem of Melkman [6], see also Karlin and

Pinkus [4], which is stated as Theorem 5.1 below for easy reference. A set of boundary forms

$$\sum_{j=1}^{m} e_{ij}D^{j-1}u(0) + \sum_{j=1}^{m} f_{ij}D^{j-1}u(1), \qquad i = 1,\dots, k$$

is said to satisfy *Postulate* J if

(i) E and F are $k \times m$ with $k \leq \min(2m, m + r)$.

(ii) the $k \times 2m$ matrix $D = \| d_{ij} \|$, where

$$\begin{aligned} d_{ij} &= e_{ij}(-1)^{j+m+r} & i &= 1,\dots, k;\ j = 1,\dots, m, \\ &= f_{i,2m+1-j} & i &= 1,\dots, k;\ j = m + 1,\dots, 2m, \end{aligned}$$

has rank k and is *sign consistent of order* k (SC_k), i.e., all nonzero subdeterminants of D have the same sign.

THEOREM 5.1. *Let the knots* $\{\xi_k\}_{k=1}^{r}$, $0 < \xi_1 < \cdots < \xi_r < 1$, *be fixed. Given points of interpolation*, $0 < x_1 < \cdots < x_{n+r-k} < 1$, *associated real data* $\{e_i\}_{i=1}^{k}$, $\{y_i\}_{i=1}^{n+r-k}$, *boundary conditions*,

$$\sum_{j=1}^{n} a_{ij}D^{j-1}S(0) + \sum_{j=1}^{n} b_{ij}D^{j-1}S(1) = e_i, \qquad i = 1,\dots, k$$

which satisfy Postulate J, *and interpolation conditions*

$$S(x_i) = y_i \qquad i = 1,\dots, n + r - k.$$

there exists a unique spline $S(x)$ *of degree* $n - 1$ *with knots* $\{\xi_k\}_{k=1}^{r}$ *satisfying these boundary and interpolation conditions iff for some* s, $0 \leq s \leq k$, *there exists a collection of indices* $i_1 < \cdots < i_s$, $(1 \leq i_\nu \leq n)$, *and* $j_1 < \cdots < j_{k-s}$, $(n + 1 \leq j_\mu \leq 2n)$ *for which*

$$D\begin{pmatrix} 1,\dots & & , k \\ i_1,\dots, i_s, j_1,\dots, j_{k-s} \end{pmatrix} \neq 0$$

while the sets $\{x_\mu\}$, $\{\xi_\nu\}$, $\{i_\alpha\}$, $\{j_\beta\}$ *satisfy*

(i) *if* $k \geq s + r$, *then*

$$x_{\nu-s} < \xi_\nu, \qquad \nu = s + 1,\dots, r,$$

$$2n + 1 - j_{k-s+1-\mu} \leq i'_{\mu+n+r-k}, \qquad \mu = 1,\dots, k - r - s.$$

where $\{i_l'\}_1^{n-s}$ *is complementary to* $\{i_l\}_1^{s}$ *in* $\{1,\dots, n\}$;

(ii) *if* $k < s + r$, *then*

$$x_{\nu-s} < \xi_\nu < x_{n-s+\nu} \qquad \nu = 1,\dots, r.$$

In (i) and (ii) the conditions are to apply when the subscripts are meaningful.

In the application of Theorem 5.1 to the case at hand, the matrix of the boundary forms (5.2) has the special form

$$\begin{Vmatrix} A_{p\times n} & 0 & B_{p\times n} & 0 \\ 0 & A^+_{(2n-p)\times n} & 0 & B^+_{(2n-p)\times n} \end{Vmatrix}.$$

Use of Laplace's expansion and some elementary simplifications show that boundary conditions (5.2) satisfy Postulate J iff the matrices $E = \| e_{ij} \|$ and $F = \| f_{ij} \|$ are SC_p and SC_{2n-p} of full rank, respectively, where

$$\begin{aligned} e_{ij} &= a_{ij}(-1)^{j+r+p}, & i &= 1,\dots, p; j = 1,\dots, n, \\ &= b_{i,2n+1-j}\,, & i &= 1,\dots, p; j = n+1,\dots, 2n, \end{aligned} \tag{5.3}$$

and

$$\begin{aligned} f_{ij} &= a^*_{ij}(-1)^{r+n}, & i &= p+1,\dots, 2n; j = 1,\dots, n, \\ &= b^*_{i,2n+1-j}(-1)^{2n+1-j}, & i &= p+1,\dots, 2n; j = n+1,\dots, 2n. \end{aligned} \tag{5.4}$$

(For convenience the rows of F are labeled $p+1,\dots, 2n$.) In fact, if D is the matrix constructed as in Postulate J for boundary conditions (5.2), then the only possibly nonzero subdeterminants of D are

$$\begin{aligned} &D\begin{pmatrix} 1,\dots & & & & & & & , 2n \\ \alpha_1,\dots, \alpha_t, & \beta_1,\dots, \beta_u, & \gamma_1,\dots, \gamma_v, & \delta_1,\dots, \delta_w \end{pmatrix} \\ &\quad = \epsilon E\begin{pmatrix} 1,\dots & & , p \\ \alpha_1,\dots, \alpha_t, & \delta_1 - 2n,\dots, \delta_w - 2n \end{pmatrix} \\ &\quad\quad \times F\begin{pmatrix} p+1,\dots & & , 2n \\ \beta_1 - n,\dots, \beta_u - n, & \gamma_1 - n,\dots, \gamma_v - n \end{pmatrix} \end{aligned} \tag{5.5}$$

where

$$\begin{gathered} 1 \leqslant \alpha_1 < \cdots < \alpha_t \leqslant n < \beta_1 < \cdots < \beta_u \leqslant 2n < \gamma_1 < \cdots < \gamma_v \\ \leqslant 3n < \delta_1 < \cdots < \delta_w \leqslant 4n, \\ t + w = p, \; u + v = 2n - p, \end{gathered}$$

and ϵ is a nonzero numerical factor whose sign is independent of the column indices. (The factor ϵ involves products of $1/w_j(x)$ for $x = 0$ and 1, apart from a factor ± 1.)

The next theorem, of general importance for boundary value problems, reveals that sign consistency assumptions on a set of boundary conditions implies analogous sign consistency for the adjoint boundary conditions. It will be used to show that F is automatically SC_{2n-p} with full rank whenever E is SC_p with full rank. This fact is of evident practical importance for determining when boundary conditions (5.2) satisfy Postulate J.

THEOREM 5.2. *Assume that the boundary forms*

$$U_i(u) = \sum_{j=1}^{n} a_{ij} D^{j-1} u(0) + \sum_{j=1}^{n} b_{ij} D^{j-1} u(1), \qquad i = 1,\ldots, p$$

are such that the matrix $D_1 = \| d_{ij}^{(1)} \|$ *is* SC_p *of full rank where*

$$\begin{aligned} d_{ij}^{(1)} &= a_{ij}(-1)^{j+r+n}, & i &= 1,\ldots, p;\ j = 1,\ldots, n. \\ &= b_{i,2n+1-j}, & i &= 1,\ldots, p;\ j = n+1,\ldots, 2n. \end{aligned}$$

Then the adjoint boundary forms

$$U_i(v) = \sum_{j=1}^{n} a_{ij}^* D^{*^{j-1}} v(0) + \sum_{j=1}^{n} b_{ij}^* D^{*^{j-1}} v(1), \qquad i = p+1,\ldots, 2n$$

constructed in (2.5) *determine a matrix* $D_{*_1} = \| d_{*_{ij}}^{(1)} \|$ *which is* SC_{2n-p} *of full rank where*

$$\begin{aligned} d_{*_{ij}}^{(1)} &= a_{ij}^*(-1)^{r+p}, & i &= p+1,\ldots, 2n;\ j = 1,\ldots, n, \\ &= b_{i,2n+1-j}^*(-1)^{2n+1-j}, & i &= p+1,\ldots, 2n;\ j = n+1,\ldots, 2n. \end{aligned}$$

Proof. Let $V = (\hat{C}^{-1})^* S^*$ denote the matrix in (2.5). Then,

$$a_{ij}^* = v_{ij}, \qquad i = p+1,\ldots, 2n;\ j = 1,\ldots, n$$

and

$$b_{ij}^* = v_{i,j+n}, \qquad i = p+1,\ldots, 2n;\ j = 1,\ldots, n.$$

Let $1 \leqslant j_1 < \cdots < j_s \leqslant n < k_{s+1} < \cdots < k_{2n-p} \leqslant 2n$. Then,

$$\begin{aligned} \Delta = D_{*_1}\begin{pmatrix} p+1,\ldots & & , 2n \\ j_1,\ldots, j_s, k_{s+1},\ldots, k_{2n-p} \end{pmatrix} &= (-1)^{s(r+p)+(2n-p-s)+(k_{s+1}+\cdots+k_{2n-p})} \\ &\times V\begin{pmatrix} p+1,\ldots & & , 2n \\ j_1,\ldots, j_s, 3n+1-k_{s+1},\ldots, 3n+1-k_{2n-p} \end{pmatrix}. \end{aligned}$$

By the Cauchy–Binet formula

$$V\begin{pmatrix} p+1,\ldots & & & & , 2n \\ j_1,\ldots,j_s, & 3n+1-k_{s+1},\ldots, & 3n+1-k_{2n-p} \end{pmatrix}$$

$$= \sum_{1\leqslant\alpha_1<\cdots<\alpha_{2n-p}\leqslant 2n} (\hat{C}^{-1})^* \begin{pmatrix} p+1,\ldots,2n \\ \alpha_1,\ldots,\alpha_{2n-p} \end{pmatrix}$$

$$\times S^* \begin{pmatrix} \alpha_1,\ldots & & , \alpha_{2n-p} \\ j_1,\ldots,j_s, & 3n+1-k_{s+1},\ldots, & 3n+1-k_{2n-p} \end{pmatrix}$$

$$= (\hat{C}^{-1})^* \begin{pmatrix} p+1,\ldots & & & , 2n \\ n+1-j_s,\ldots, & n+1-j_1, & k_{s+1},\ldots, & k_{2n-p} \end{pmatrix}$$

$$\times \prod_{l=1}^{s} \frac{(-1)}{w_{n+1-j_l}(0)} \prod_{l=s+1}^{2n-p} \frac{1}{w_{k_l-n}(1)} (-1)^{s(s-1)/2},$$

because of the special form of S^*. Thus,

$$\Delta = (-1)^{s(r+p)+(2n-p)+s(s-1)/2-(k_{s+1}+\cdots+k_{2n-p})}$$

$$\times \prod_{l=1}^{s} \frac{1}{w_{n+1-j_l}(0)} \prod_{l=s+1}^{2n-p} \frac{1}{w_{k_l-n}(1)}$$

$$\times \hat{C}^{-1} \begin{pmatrix} n+1-j_s,\ldots, & n+1-j_1, & k_{s+1},\ldots, & k_{2n-p} \\ p+1,\ldots & & & , 2n \end{pmatrix}$$

But (see [1, p. 3]),

$$\hat{C}^{-1} \begin{pmatrix} n+1-j_s,\ldots, & n+1-j_1, & k_{s+1},\ldots, & k_{2n-p} \\ p+1,\ldots & & & , 2n \end{pmatrix}$$

$$= (-1)^{(2n-p)(2n+p+1)/2+(n+1)s-(j_1+\cdots+j_s)+(k_{s+1}+\cdots+k_{2n-p})}$$

$$\times \hat{C} \begin{pmatrix} 1,\ldots & & & , p \\ n+1-j'_{n-s},\ldots, & n+1-j_1', & k_1',\ldots, & k'_{p+s-n} \end{pmatrix} \Big/ \det \hat{C},$$

where

$$1 \leqslant j_1' < \cdots < j'_{n-s} \leqslant n \text{ is complementary to } \{j_l\}_1^s \text{ in } \{1,\ldots,n\},$$

$$n+1 \leqslant k_1' < \cdots < k'_{p+s-n} \leqslant 2n \text{ is complementary to } \{k_l\}_{s+1}^{2n-p} \text{ in } \{n+1,\ldots,2n\}.$$

So

$$\Delta = (-1)^{s(r+p)+(2n-p)(2n+p+3)/2+s(s-1)/2+(n+1)s-(j_1+\cdots+j_s)}$$

$$\times \prod_{l=1}^{s} \frac{1}{w_{n+1-j_l}(0)} \prod_{l=s+1}^{2n-p} \frac{1}{w_{k_l-n}(1)} \frac{1}{\det \hat{C}}$$

$$\times \hat{C}\begin{pmatrix} 1, \ldots & & & & , p \\ n+1-j'_{n-s}, \ldots, & n+1-j_1', & k_1', \ldots, & k'_{p+s-n} \end{pmatrix}.$$

Next a short calculation shows that

$$C\begin{pmatrix} 1, \ldots & & & , p \\ n+1-j'_{n-s}, \ldots, n+1-j_1', & k_1', \ldots, & k'_{p+s-n} \end{pmatrix}$$

$$= (-1)^{(n-s)(r+1)-(j_1'+\cdots+j'_{n-s})+(p+s-n-1)(p+s-n)/2}$$

$$\times D_1\begin{pmatrix} 1, \ldots & & , p \\ n+1-j'_{n-s}, \ldots, n+1-j_1', & & \\ & 3n+1-k'_{p+s-n}, \ldots, & 3n+1-k_1' \end{pmatrix}.$$

Thus

$$\Delta = (-1)^{(2n-p)(2n+p+3)/2+(n)(n+1)/2}$$

$$\times (-1)^{s(r+p)+s(s-1)/2+(n+1)s+(n-s)(r+1)+(s+p-n-1)(s+p-n)/2}$$

$$\times \prod_{l=1}^{s} \frac{1}{w_{n+1-j_l}(0)} \prod_{l=s+1}^{2n-p} \frac{1}{w_{k_l-n}(1)} \frac{1}{\det \hat{C}}$$

$$\times D_1\begin{pmatrix} 1, \ldots & & , p \\ n+1-j'_{n-s}, \ldots, n+1-j_1' & & \\ & 3n+1-k'_{p+s-n}, \ldots, & 3n+1-k_1' \end{pmatrix}.$$

The second factor on the right simplifies to

$$(-1)^{n(r+1)+(p-n-1)(p-n)/2}.$$

Combining this with the first factor in the preceding equation for Δ and simplifying yields

$$(-1)^{p(r+p)}.$$

Thus

$$D_{*1}\begin{pmatrix} p+1,\ldots & & , 2n \\ j_1,\ldots,j_s, & k_{s+1},\ldots, & k_{2n-p} \end{pmatrix}$$

$$= (-1)^{n(r+p)} \prod_{l=1}^{s} \frac{1}{w_{n+1-j_l}(0)} \prod_{l=s+1}^{2n-p} \frac{1}{w_{k_l-n}(1)} \cdot \frac{1}{\det \hat{C}}$$

$$\times D_1 \begin{pmatrix} 1,\ldots & & , p \\ n+1-j'_{n-s},\ldots, n+1-j_1', & & \\ & 3n+1-k'_{p+s-n},\ldots, 3n+1-k_1' & \end{pmatrix}$$

which proves the theorem. ∎

Remark 5.1. The proof actually shows that D_1 is SC_p of full rank iff D_{*_1} is SC_{2n-p} of full rank.

Theorem 5.2 yields

THEOREM 5.3. *Boundary conditions* (5.2) *satisfy Postulate* J *iff the matrix* E *in* (5.3) *is* SC_p *of full rank.*

Proof. The definitions of E, F, D_1, and D_{*_1}, yield

$$E\begin{pmatrix} 1,\ldots & & & , p \\ n+1-j'_{n-s},\ldots, n+1-j_1', & 3n+1-k'_{p-s-n},\ldots, & 3n+1-3k_1' \end{pmatrix}$$

$$= (-1)^{(n-s)(p+n)} D_1 \begin{pmatrix} 1,\ldots & & , p \\ n+1-j'_{n-s},\ldots, n+1-j_1', & & \\ & 3n+1-k'_{p+s-n},\ldots, 3n+1-k_1' & \end{pmatrix}$$

and

$$F\begin{pmatrix} p+1,\ldots & & , 2n \\ j_1,\ldots,j_s, & k_{s+1},\ldots, & k_{2n-p} \end{pmatrix} = (-1)^{s(n+p)} D_{*1}\begin{pmatrix} p+1,\ldots & & , 2n \\ j_1,\ldots,j_s, & k_{s+1},\ldots, & k_{2n-s} \end{pmatrix}.$$

The last equation in the proof of Theorem 5.2 can now be expressed as

$$F\begin{pmatrix} p+1,\ldots & & , 2n \\ j_1,\ldots,j_s, & k_{s+1},\ldots, & k_{2n-p} \end{pmatrix}$$

$$= \frac{(-1)^{n(r+n)}}{\det \hat{C}} \prod_{l=1}^{s} \frac{1}{w_{n+1-j_l}(0)} \prod_{l=s+1}^{2n-p} \frac{1}{w_{k_l-n}(1)}$$

$$\times E \begin{pmatrix} 1,\ldots & & , p \\ n+1-j'_{n-s},\ldots, n+1-j_1', & & \\ & 3n+1-k'_{p+s-n},\ldots, 3n+1-k_1' & \end{pmatrix}.$$

Thus E is SC_p of full rank iff F is SC_{2n-p} of full rank. The theorem follows from the remarks preceding (5.3). ∎

Application of this theorem in conjunction with Theorem 5.1 yields

THEOREM 5.4. *Let E and F be given by* (5.3) *and* (5.4). *Assume E is SC_p of full rank. Then there exists a unique monospline $\tilde{N} \in \mathcal{M}_{2n,r}$ satisfying*

$$\begin{aligned} U(\tilde{N}) &= 0, \\ U^*(L\tilde{N}) &= 0, \\ \tilde{N}(\xi_k) &= 0, \qquad k = 1,\dots, r, \end{aligned} \tag{5.6}$$

iff there exist indices $\{\alpha_a\}$, $\{\beta_b\}$, $\{\gamma_c\}$, $\{\delta_d\}$ such that

$$E\begin{pmatrix} 1,\dots & & , p \\ \alpha_1,\dots,\alpha_t, & \delta_1 - 2n,\dots, & \delta_w - 2n \end{pmatrix} \neq 0,$$

$$F\begin{pmatrix} p+1,\dots & & , 2n \\ \beta_1 - n,\dots,\beta_u - n, & \gamma_1 - n,\dots, & \gamma_v - n \end{pmatrix} \neq 0,$$

where $t + w = p$, $u + v = 2n - p$, and the indices

$$\begin{aligned} 1 &\leqslant \alpha_1 < \cdots < \alpha_t \leqslant n < \beta_1 < \cdots < \beta_u \\ &\leqslant 2n < \gamma_1 < \cdots < \gamma_v \leqslant 3n < \delta_1 < \cdots < \delta_w \leqslant 4n \end{aligned}$$

must in addition satisfy: Let $\{i_1,\dots,i_s\} = \{\alpha_1,\dots,\alpha_t, \beta_1,\dots,\beta_u\}$ so $s = t + u$ and $\{j_1,\dots,j_{2n-s}\} = \{\gamma_1,\dots,\gamma_v, \delta_1,\dots,\delta_w\}$.

Case 1. *Assume $r = 0$, i.e., no knots occur. Then the indices must satisfy*

$$4n + 1 - j_{2n+1-s-\mu} \leqslant i_\mu', \qquad \mu = 1,\dots, 2n - s.$$

Case 2. *Assume $r \geqslant 1$. Then the indices must be such that $s > 0$ and*

(i) *if* $2n > s + r$, $4n + 1 - j_{2n+1-s-\mu} \leqslant i'_{\mu+r}$, $\mu = 1,\dots, 2n - r - s$, *while*

(ii) *if $2n < s + r$, then $s < 2n$.*

Here $\{i_l'\}_1^{2n-s}$ is the complementary set of indices to $\{i_l\}_1^s$ in $\{1,\dots, 2n\}$, and the above conditions are to apply only when the subscripts are meaningful.

Proof. With the sets $\{\alpha_a\}$, $\{\beta_b\}$, $\{\gamma_c\}$, $\{\delta_d\}$, $\{i_l\}$, and $\{j_l\}$ defined as above, (5.5) shows that

$$D\begin{pmatrix} 1,\dots & & & , 2n \\ \alpha_1,\dots,\alpha_t, & \beta_1,\dots,\beta_u, & \gamma_1,\dots,\gamma_v, & \delta_1,\dots,\delta_w \end{pmatrix} \neq 0$$

iff both determinants listed in the theorem are nonzero. By Theorem 5.1 there exists a unique monospline $\tilde{N}$ satisfying (5.6) iff these determinants are nonzero for indices $\{i_l\}$ and $\{j_l\}$ satisfying the stated conditions. (Note that the knots and points of interpolation agree in this case.) ▌

Remark 5.2. If $p = 0$ (resp., $p = 2n$) the condition on E (resp., F) is to be dropped.

Remark 5.3. The adjoint boundary forms and hence F must be known if $\tilde{N}$ is to be computed explicitly; however, the conditions in the theorem guaranteeing the existence of $\tilde{N}$ can be stated without explicit reference to the adjoint boundary forms. Indeed, by the last equation in the proof of Theorem 5.3, the condition on F is equivalent to

$$E\begin{pmatrix} 1, \dots & & & & , p \\ 2n+1-\beta'_{n-u}, \dots, & 2n+1-\beta_1', & 4n+1-\gamma'_{n-r}, \dots, & 4n+1-\gamma_1' \end{pmatrix} \neq 0$$

where $\{\beta_l'\}_1^{n-u}$ is complementary to $\{\beta_l\}_1^u$ in $\{n+1, \dots, 2n\}$, and $\{\gamma_l'\}_1^{n-r}$ is complementary to $\{\gamma_l\}_1^v$ in $\{2n+1, \dots, 3n\}$.

Remark 5.4. It is interesting to consider the example in Section 4 involving the midpoint rule in the context of Theorem 5.4. In this case $p = 0$ and F is SC_4 of full rank. Since $r = 1$, Case 2 of the theorem is relevant. It is easily checked that Case 2(i) applies and that the index requirement is not satisfied.

EXAMPLES. (a) *Periodic boundary forms.* Because of their frequent occurrence in applications, periodic boundary forms are among the most important mixed boundary forms. In this case the matrix D itself is readily available

$$D = \begin{Vmatrix} (-1)^{r+1} & & & & & & -1 \\ & (-1)^{r+2} & & & & 1 & \\ & & \ddots & & \cdot^{\cdot^{\cdot}} & & \\ & & & (-1)^r \quad -1 & & & \end{Vmatrix}_{2n \times 4n}$$

assuming that all the functions $w_j(x)$ are periodic with period 1 because then the boundary forms are self-adjoint. It is easily verified that D is SC_{2n} iff r is *odd* in which case

$$D\begin{pmatrix} 1, \dots & & , 2n \\ i_1, \dots, i_s, j_1, \dots, j_{2n-s} \end{pmatrix} = \begin{cases} (-1)^n, & i_l' = 4n + 1 - j_{2n+1-s-l}, \\ 0 & \text{otherwise}, \end{cases}$$

where $l = 1, \dots, 2n-s$, $\{i_l'\}_1^{2n-s}$ is complementary to $\{i_l\}_1^s$ in $\{1, \dots, 2n\}$. and $1 \leqslant i_1 < \cdots < i_s \leqslant 2n < j_1 < \cdots < j_{2n-s} \leqslant 4n$. Since r is odd, Case 2 of the theorem is relevant. If $1 \leqslant r < 2n$, choose $s = 2n - r$. Then $s > 0$ and

$s + r = 2n$. If $r \geqslant 2n$ (hence, $r > 2n$) choose $s = 1$. In either case all the stipulations of Case 2 are met and $\tilde{N}$ is uniquely determined.

If the $w_j(x)$ are not periodic, calculation of the adjoint boundary forms shows that the matrix D has positive factors multiplying the columns $n + 1,\ldots, 3n$ above, and so the previous analysis can be carried out with inessential changes. Alternatively, the adjoint boundary forms need not be calculated at all in view of Remark 5.3. For periodic boundary forms the matrix E is

$$\left\| \begin{array}{cccccc} (-1)^{r+n+1} & & & & & -1 \\ & (-1)^{r+n+2} & & & -1 & \\ & & \ddots & \cdot^{\cdot^{\cdot}} & & \\ & & (-1)^{r} \quad -1 & & & \end{array} \right\|_{n \times 2n}$$

and E is SC_n iff r is *odd* in which case

$$E\begin{pmatrix} 1,\ldots & & , n \\ \alpha_1 ,\ldots, \alpha_t , \delta_1 - 2n,\ldots, \delta_w - 2n \end{pmatrix} = \begin{cases} (-1)^{n(n+1)/2}, & \alpha_l' = 4n + 1 - \delta_{w+1-l} , \\ 0 & \text{otherwise}, \end{cases}$$

for $l = 1,\ldots, w$. Likewise,

$$E\begin{pmatrix} 1,\ldots & & & , n \\ 2n + 1 - \beta_{n-u}' ,\ldots, 2n + 1 - \beta_1', 4n + 1 - \gamma_{n-v}' ,\ldots, 4n + 1 - \gamma_1' \end{pmatrix}$$
$$= \begin{cases} (-1)^{n(n+1)/2}, & \beta_l' = 4n + 1 - \gamma_{v+1-l} , \\ 0, & \text{otherwise}, \end{cases}$$

for $l = 1,\ldots, v$. Thus to obtain nonzero values for the appropriate E subdeterminants

$$\alpha_l' = 4n + 1 - \delta_{w+1-l} , \qquad l = 1,\ldots, w,$$
$$\beta_l' = 4n + 1 - \gamma_{v+1-l} , \qquad l = 1,\ldots, v.$$

If $\{i_1 ,\ldots, i_s\} = \{\alpha_1 ,\ldots, \alpha_t , \beta_1 ,\ldots, \beta_u\}$ and $\{ j_1 ,\ldots, j_{2n-s}\} = \{\gamma_1 ,\ldots, \gamma_v , \delta_1 ,\ldots, \delta_w\}$ the previous stipulations are

$$i_l' = 4n + 1 - j_{2n-s+1-l} \qquad l = 1,\ldots, 2n - s.$$

Now it follows exactly as before that $\tilde{N}$ is uniquely determined.

(b) *Antiperiodic boundary forms.* These forms can be treated in the same manner as periodic forms. In this case, r must be *even* for D to be SC_{2n} .

6. Separated Boundary Forms

Theorem 5.2 allows us to refine some of the results in [3] where the boundary forms are separated. In this case Postulate J is equivalent (see [4]) to Postulate I stated below.

Assume separated boundary forms,

$$U_i(u) = \sum_{j=1}^{n} a_{ij} D^{j-1} u(0), \qquad i = 1, \ldots, p,$$
$$U_{i+p}(u) = \sum_{j=1}^{n} b_{ij} D^{j-1} u(1), \qquad i = 1, \ldots, q, \tag{6.1}$$

and let

$$A = \| a_{ij} \|, \qquad B = \| b_{ij} \| .$$

These boundary forms are assumed to satisfy

Postulate I.

(i) $p + q \leqslant 2n$;

(ii) $\tilde{A} = \| a_{ij}(-1)^j \|$ is SC_p with rank p;

(iii) B is SC_q with rank q.

Notice that the rank conditions imply $p \leqslant n$ and $q \leqslant n$.

Theorem 6.1. *Let the boundary forms* (6.1) *satisfy Postulate* I. *Then adjoint boundary forms can be constructed in* (2.5) *to have the form*

$$U_i^*(u) = \sum_{j=1}^{n} a_{ij}^* D^{*^{j-1}} u(0), \qquad i = 1, \ldots, n-p,$$

$$U_{i+n-p}^*(u) = \sum_{j=1}^{n} b_{ij}^* D^{*^{j-1}} u(1), \qquad i = 1, \ldots, n-q,$$

and satisfy

$$A_* = \| a_{ij}^* \| \text{ is } SC_{n-p} \text{ with rank } n-p,$$
$$\tilde{B}_* = \| b_{ij}^*(-1)^j \| \text{ is } SC_{n-q} \text{ with rank } n-q,$$

where $B_* = \| b_{ij}^* \|$.

Proof. Using notation similar to that in Section 2, if

$$C = \begin{Vmatrix} A & 0 \\ 0 & B \end{Vmatrix}_{(p+q)\times 2n}$$

then there are matrices $\hat{A}$ and $\hat{B}$ of order $(n-p) \times n$ and $(n-q) \times n$, respectively, such that

$$\hat{C} = \begin{Vmatrix} A & 0 \\ 0 & B \\ \hat{A} & 0 \\ 0 & \hat{B} \end{Vmatrix}_{2n \times 2n}$$

is nonsingular because A and B have full rank. There is a permutation matrix P such that

$$P\hat{C} = \begin{Vmatrix} A_1 & 0 \\ 0 & B_1 \end{Vmatrix}_{2n \times 2n}$$

where

$$A_1 = \begin{Vmatrix} A \\ \hat{A} \end{Vmatrix}_{n \times n} \qquad B_1 = \begin{Vmatrix} B \\ \hat{B} \end{Vmatrix}_{n \times n}$$

and A_1 and B_1 are nonsingular. Thus,

$$\hat{C}^{-1}P^* = \begin{Vmatrix} A_1^{-1} & 0 \\ 0 & B_1^{-1} \end{Vmatrix},$$

$$P(\hat{C}^{-1})^* = \begin{Vmatrix} (A_1^{-1})^* & 0 \\ 0 & (B_1^{-1})^* \end{Vmatrix}.$$

With S defined as in Section 2, it follows that

$$P(\hat{C}^{-1})^*S^* = \begin{Vmatrix} A_2 & 0 \\ 0 & B_2 \end{Vmatrix}_{2n \times 2n}$$

for certain $n \times n$ matrices A_2 and B_2. Consequently, the matrix $(\hat{C}^{-1})^*S^*$ used to construct the adjoint boundary forms yields separated boundary forms of the type stated in the theorem.

In the context of Theorem 5.2 for separated boundary conditions $D_1 = \| d_{ij}^{(1)} \|$ where,

$$\begin{aligned} d_{ij}^{(1)} &= a_{ij}(-1)^{j+r+n}, && i = 1,\ldots, p;\ j = 1,\ldots, n, \\ &= b_{i-p,2n+1-j}, && i = p+1,\ldots, p+q;\ j = n+1,\ldots, 2n, \\ &= 0, && \text{otherwise} \end{aligned}$$

An elementary linear dependence argument (see [4]) reveals that the only possible nonzero subdeterminants of D_1 have the form

$$D_1 \begin{pmatrix} 1,\ldots & & ,p+q \\ j_1,\ldots, j_p, n+k_1,\ldots, n+k_q \end{pmatrix} \tag{6.2}$$

for $1 \leqslant j_1 < \cdots < j_p \leqslant n$ and $1 \leqslant k_1 < \cdots < k_q \leqslant n$. The determinant (6.2) is easily seen to be equal to

$$(-1)^{p(r+n)+(q-1)q/2} \tilde{A}\begin{pmatrix} 1,\ldots, p \\ j_1, \ldots, j_p \end{pmatrix} B\begin{pmatrix} 1,\ldots & & , q \\ n+1-k_q, \ldots, n+1-k_1 \end{pmatrix}.$$

Hence D_1 is SC_{p+q} of full rank iff the boundary forms (6.1) satisfy Postulate I. Entirely similar reasoning confirms that, for the case at hand, the matrix D_{*1} in Theorem 5.2 is $\mathrm{SC}_{2n-(p+q)}$ of full rank iff the matrices A_* and $\tilde{B}_*$ defined in the theorem are, respectively, SC_{n-p} and SC_{n-q} of full rank. Now Theorem 5.2 implies the desired result. ▮

Theorem 6.1 implies the following strengthened version of Theorem 3.2 in [3].

THEOREM 6.2. *Let the boundary forms* (6.1) *satisfy Postulate* I. *Then there is a unique monospline* $\tilde{N} \in \mathscr{M}_{2n,r}$ *satisfying*

$$U(\tilde{N}) = 0,$$
$$U^*(L\tilde{N}) = 0,$$
$$\tilde{N}(\xi_k) = 0, \qquad k = 1,\ldots, r,$$

iff there are indices $1 \leqslant i_1 < \cdots < i_p \leqslant n < i_{p+1} < \cdots < i_n \leqslant 2n$, $1 \leqslant j_1 < \cdots < j_q \leqslant n < j_{q+1} < \cdots < j_n \leqslant 2n$ *such that*

$$A\begin{pmatrix} 1,\ldots, p \\ i_1, \ldots, i_p \end{pmatrix} \neq 0, \qquad A_*\begin{pmatrix} 1,\ldots & , n-p \\ i_{p+1}-n, \ldots, i_n-n \end{pmatrix} \neq 0,$$
$$B\begin{pmatrix} 1,\ldots, q \\ j_1, \ldots, j_q \end{pmatrix} \neq 0, \qquad B_*\begin{pmatrix} 1,\ldots & , n-q \\ j_{q+1}-n, \ldots, j_n-n \end{pmatrix} \neq 0, \tag{6.3}$$

and if $n > r$,

$$j_\mu \leqslant i'_{\mu+r}, \qquad \mu = 1,\ldots, n-r \tag{6.4}$$

where $\{i_l'\}_1^n$ *is complementary to* $\{i_l\}_1^n$ *in* $\{1,\ldots, 2n\}$. *Consequently, if* $r \geqslant n$, $\tilde{N}$ *is always uniquely determined* (cf. *Theorem* 4.3).

Proof. The matrix of the boundary conditions which $\tilde{N}$ must satisfy at $x = 0$ is (cf. (5.2)),

$$A_1 = \left\| \begin{matrix} \| a_{ij} \|_{\substack{i=1,\ldots,p \\ j=1,\ldots,n}} & 0_{p\times n} \\ 0_{(n-p)\times n} & \| (-1)^{j-1} a^*_{ij} / w_{n+2-j}(0) \|_{\substack{i=1,\ldots,n-p \\ j=1,\ldots,n}} \end{matrix} \right\|.$$

The matrix $\tilde{A}_1$ constructed as in Postulate I is

$$\begin{Vmatrix} \|(-1)^j a_{ij}\| & 0 \\ 0 & \|(-1)^{n-1} a_{ij}^*/w_{n+2-j}(0)\| \end{Vmatrix}.$$

Evidently, $\tilde{A}_1$ is SC_n with rank n iff $\tilde{A}$ is SC_p with rank p and A_* is SC_{n-p} with rank $n-p$. The matrix of the boundary conditions at $x = 1$ is

$$B_1 = \begin{Vmatrix} \|b_{ij}\|_{\substack{i=1,\ldots,q \\ j=1,\ldots,n}} & 0_{q\times n} \\ 0_{(n-q)\times n} & \|b_{ij}^*(-1)^{j-1}/w_{n+2-j}(1)\|_{\substack{i=1,\ldots,n-q \\ j=1,\ldots,n}} \end{Vmatrix}$$

which is SC_n with rank n iff B is SC_q with rank q and $\tilde{B}_*$ is SC_{n-q} with rank $n-q$. By assumption $\tilde{A}$ and B are, respectively, SC_{n-p} and SC_{n-q} of full rank. Thus, the matrices A_1 and B_1 satisfy Postulate I with n replaced by $2n$, p replaced by n, and q replaced by n. The theorem now follows by invoking the basic interpolation result, Theorem 2 in [2], for the case at hand. ∎

Remark 6.1. The index restriction (6.4) was inadvertently omitted in Theorem 3.2 in [3].

Remark 6.2. Just as for mixed boundary forms (see Remark 5.3) it is possible to express conditions (6.3) in Theorem 6.2 without explicit reference to the adjoint boundary forms. Specifically, the inequalities involving A_* and B_* are equivalent to

$$A\begin{pmatrix} 1,\ldots & & ,p \\ 2n+1-i_n',\ldots, & 2n+1-i_{n-p+1}' \end{pmatrix} \neq 0, \tag{6.5}$$

$$B\begin{pmatrix} 1,\ldots & & ,q \\ 2n+1-j_n',\ldots, & 2n+1-j_{n-q+1}' \end{pmatrix} \neq 0, \tag{6.6}$$

where $\{i_l'\}_1^n$ and $\{j_l'\}_1^n$ are complementary, respectively, to $\{i_l\}_1^n$ and $\{j_l\}_1^n$ in $\{1,\ldots, 2n\}$.

The following result generalizes Theorem 1 in [8]. It applies in particular to boundary forms encountered in the study of vibrating physical systems.

THEOREM 6.3. *Consider quadrature formulas of the form,*

$$Q(u) = \sum_{i=1}^{p+q} a_i U_i(u) + \sum_{k=1}^{r} c_k u(\xi_k) \tag{6.7}$$

with boundary forms (6.1) *satisfying Postulate* I *and the requirements*

$$A\begin{pmatrix} 1,\ldots, p \\ 1,\ldots, p \end{pmatrix} \neq 0, \tag{6.8}$$

$$B\begin{pmatrix} 1,\ldots, q \\ 1,\ldots, q \end{pmatrix} \neq 0. \tag{6.9}$$

Let $\mathscr{C}$ be the class of quadrature formulas (6.7) *which are exact on L-polynomials. Then $\mathscr{C}$ is nonempty iff there exists a monospline $\tilde{N} \in \mathscr{M}_{2n,r}$ satisfying*

$$\begin{aligned} U(\tilde{N}) &= 0, \\ U^*(L\tilde{N}) &= 0, \\ \tilde{N}(\xi_k) &= 0, \qquad k = 1;..., r. \end{aligned} \tag{6.10}$$

Furthermore, $\tilde{N}$ is uniquely determined by these requirements iff

$$r \geqslant n - (p + q).$$

Proof. Let

$$\begin{aligned} \{i_l\}_1^n &= \{1,..., p, n + 1,..., 2n - p\}, \\ \{j_l\}_1^n &= \{1,..., q, n + 1,..., 2n - q\}. \end{aligned} \tag{6.11}$$

Then

$$\begin{aligned} \{i_l'\}_1^n &= \{p + 1,..., n, 2n - p + 1,..., 2n\}, \\ \{j_l'\}_1^n &= \{q + 1,..., n, 2n - q + 1,..., 2n\}. \end{aligned} \tag{6.12}$$

Consequently,

$$A\begin{pmatrix} 1,..., p \\ i_1, ..., i_p \end{pmatrix} = A\begin{pmatrix} 1,... & , p \\ 2n + 1 - i_n',..., 2n + 1 - i'_{n-p+1} \end{pmatrix} = A\begin{pmatrix} 1,..., p \\ 1,..., p \end{pmatrix} \neq 0.$$

and

$$B\begin{pmatrix} 1,..., q \\ j_1, ..., j_q \end{pmatrix} = B\begin{pmatrix} 1,... & , q \\ 2n + 1 - j_n',..., 2n + 1 - j'_{n-q+1} \end{pmatrix} = B\begin{pmatrix} 1,..., q \\ 1,..., q \end{pmatrix} \neq 0.$$

In view of Remark 6.2, the requirements (6.3) of Theorem 6.2 hold for the indices (6.11). Furthermore, in the event $n > r$, there will exist indices satisfying (6.3) and (6.4) iff the indices (6.11) satisfy (6.4) because the selection (6.11) determines the smallest possible j''s and largest possible i''s. Consequently, there exists a unique monospline $\tilde{N}$ satisfying (6.10) iff either $r \geqslant n$ or, if $r < n$, the indices (6.11) satisfy (6.4).

From (6.11) and (6.12),

$$\begin{aligned} j_\mu &= \mu, & \mu &= 1,..., q, \\ &= n + \mu - q, & \mu &= q + 1,..., n, \end{aligned}$$

and

$$\begin{aligned} i_\mu' &= p + \mu, & \mu &= 1,..., n - p, \\ &= n + \mu, & \mu &= n - p + 1,..., n. \end{aligned}$$

Thus, (6.4) fails to hold iff there is an index μ such that

$$q < \mu \leq n - r.$$

and either

$$n - \mu - q > p - \mu - r. \qquad \text{if } \mu + r \leq n - p$$

or

$$n - \mu - q > n + \mu - r, \qquad \text{if } \mu + r > n - p.$$

The last requirement can never be satisfied, the two before it are equivalent to the existence of μ satisfying $q < \mu \leq n - r - p$, i.e., to $q < n - r - p$. Consequently, $\tilde{N}$ is uniquely determined iff $r \geq n - (p + q)$, which proves the last assertion in the theorem.

The analysis above establishes that the existence and uniqueness of an $\tilde{N}$ satisfying (6.10) occurs iff $r \geq n - (p + q)$. By Theorem 4.3, $\mathscr{C}$ is nonempty if an $\tilde{N}$ exists satisfying (6.10). Thus, it remains to show that $\tilde{N}$ exists satisfying (6.10) when $\mathscr{C}$ is nonempty and $n - (p + q) > r \geq 0$. Construct $\tilde{N}_1 \in \mathscr{M}_{2n,r}$ satisfying $U(\tilde{N}_1) = 0$ and $U^*(L\tilde{N}_1) = 0$ as in Remark 4.1. Fix points $0 < x_1 < \cdots < x_\lambda < 1$ and data $y_1, \ldots, y_\lambda$ where $\lambda = n - (p + q)$. By the basic interpolation result, Theorem 2 in [2], there exists a unique L-polynomial, P, such that $U(P) = 0$ and $P(x_l) = y_l$, $l = 1, \ldots, \lambda$. (Indeed, the hypotheses of that theorem are met using the indices $\{i_l\}_1^p = \{l\}_1^p$ and $\{j_l\}_1^q = \{l\}_1^q$), By further specifying $x_l = \xi_l$ and $y_l = -\tilde{N}_1(\xi_l)$ for $l = 1, \ldots, r$ (recall $r < \lambda$) it follows that $\tilde{N} = \tilde{N}_1 + P$ satisfies (6.10). ∎

Remark 6.3. Schoenberg's result, Theorem 1 in [8], is the uniqueness assertion of Theorem 6.3 when $p = q$ and the boundary forms are specified by the matrices

$$A = [I_p, 0_{n-p}], \qquad B = [I_p, 0_{n-p}],$$

where I_p is the $p \times p$ identity matrix. The uniqueness result is quite useful in the actual calculation of $\tilde{N}$; see [8].

7. Boundary Forms for Vibrating Systems

The results of Section 6 will be used to determine best quadrature formulas based on the specific boundary forms,

$$\begin{aligned} U_i(u) &= D^{i-1}u(0) + (-1)^{n+p+i-1} c_i D^{n-i}u(0), \qquad & i = 1, \ldots, p. \\ U_{p+i}(u) &= D^{i-1}u(1) + (-1)^{q+i} d_i D^{n-i}u(1), \qquad & i = 1, \ldots, q. \end{aligned} \tag{7.1}$$

where $0 \leqslant p, q \leqslant n$, $0 \leqslant c_i, d_i$. These boundary forms arise in physical oscillation problems (see [1, Chap. 10, Section 7]) and were treated in [3]. The results below sharpen Theorem 4.1 in [3]. The assumptions required on the c's and d's in (7.1) and the attendent analysis are somewhat different according as n is even or odd. Suppose n is *even*, the case of primary physical interest. (Comments on the situation when n is odd are given at the end of this section).

It was shown in [3] that the matrices A and B of the boundary forms corresponding to $x = 0$ and $x = 1$ in (7.1) are, respectively, SC_p and SC_q of full rank. (This was done by direct evaluation of the determinants in question.) Thus, the boundary forms (7.1) satisfy Postulate I and, additionally, it is easily verified that

$$A\begin{pmatrix}1,..., p\\1,..., p\end{pmatrix} \neq 0,$$

$$B\begin{pmatrix}1,..., q\\1,..., q\end{pmatrix} \neq 0,$$

because of the special form of A and B. Appeal to Theorem 6.3 yields the following refined version of Theorem 4.1 in [3].

THEOREM 7.1. *Consider the class* $\mathscr{C}$ *of quadrature formulas* (6.7) *exact on L-polynomials, with boundary forms* (7.1). *The class* $\mathscr{C}$ *is nonempty iff* (6.10) *has a solution* $\tilde{N}$. *Furthermore,* $\tilde{N}$ *is uniquely determined by* (6.10) *iff* $r \geqslant n - (p + q)$.

Remark 7.1. $\tilde{N}$ may exist, equivalently $\mathscr{C}$ may be nonempty, when $r < n - (p + q)$. The example of Section 4 using the midpoint rule is a case in point. There $n = 2$, $r = 1$, and $p = q = 0$.

Remark 7.2. The preceding discussion as well as that in [3, Section 4] is easily modified to cover the case when n is *odd*. However, it must be assumed that, for $l = [n/2] + 1$,

$$(-1)^l + (-1)^p c_l \neq 0,$$

and when $p > l$, that,

$$c_{l-j} c_{l+j} \neq 1, \qquad j = 1,..., p - l.$$

These assumptions and corresponding ones on the d's are needed to insure that the boundary forms (7.1) have full rank.

Remark 7.3. An alternative proof that the matrices A and B are SC_p and SC_q of full rank can be based on Theorem 2.2 in [1].

8. Extensions and Remarks

8.1. *Best L_2-Approximations*

The preceding results characterizing best quadrature formulas can also be viewed as characterizing the best monospline approximation to zero in $L_2[0, 1]$ among all monosplines satisfying specified boundary conditions and having prescribed knots.

8.2. *Weight Functions*

The previous results and their proofs extend immediately to quadrature formulas approximating

$$\int_0^1 u(x)\, w(x)\, dx$$

where $w(x)$ is a positive, continuous weight function. The only change necessary is to redefine ψ_n in the definition of monospline to be the unique solution to the initial value problem

$$Lu = w$$
$$D^{j-1}u(0) = 0, \qquad j = 1,..., n.$$

A definite integral representation of ψ_n is available by integration.

8.3. *Multiknot Quadrature Formulas*

The analysis of the preceding sections extends to include the important case of multiknot quadrature formulas

$$Q(u) = \sum_{i=1}^{p} a_i U_i(u) + \sum_{k=1}^{r} \sum_{j=1}^{\mu_k} c_{kj} D^{j-1} u(\xi_k) \tag{8.1}$$

which are exact on L-polynomials. Here

$$1 \leq \mu_k \leq n, \qquad k = 1,..., r$$

specifies the multiplicity of the knot ξ_k. The analog of Theorem 3.1 establishes a 1 : 1 correspondence between quadrature formulas of the form (8.1) exact on L-polynomials and L^*-monosplines of the form

$$M(x) = \psi_n^*(x) + \sum_{\nu=1}^{n} b_\nu u_\nu^*(x) + \sum_{k=1}^{r} \sum_{j=1}^{\mu_k} d_{kj} \phi_{n+1-j}^*(x; \xi_k)$$

satisfying the adjoint boundary conditions, $U^*(M) = 0$. Here ϕ_{n-j+1}^* is the fundamental solution for $D_j^* \cdots D_n^*$ constructed as in Section 2. If $Q(u)$ in (8.1) corresponds to M, then

$$
\begin{aligned}
a_i &= -U_{c,i}^*(M), && i = 1,\dots, p,\\
c_{k,j} &= \frac{D^{*n-j}M(\xi_k+) - D^{*n-j}M(\xi_k-)}{w_j(\xi_k)}, && k = 1,\dots, r;\ j = 1,\dots,\mu_k\\
R(u) &= \int_0^1 (Lu)\, M\, dx.
\end{aligned}
$$

The best quadrature formula $\tilde{Q}(u)$ is again determined by the monospline $\tilde{M}$ satisfying the adjoint boundary conditions $U^*(\tilde{M}) = 0$ and the orthogonality requirement

$$\int_0^1 \tilde{M} S\, dx = 0$$

for all L^*-splines S satisfying the same boundary conditions and with knots of multiplicity μ_k at ξ_k. It follows as for Theorem 4.2 that the class $\mathscr{C}$ of admissible quadrature formulas is nonempty if an L^*L-monospline $\tilde{N}$ with knots ξ_k of multiplicity μ_k exists satisfying

$$
\begin{aligned}
U(\tilde{N}) &= 0,\\
U^*(L\tilde{N}) &= 0, \qquad\qquad\qquad\qquad\qquad\qquad (8.2)\\
D^{j-1}\tilde{N}(\xi_k) &= 0, \qquad k = 1,\dots, r;\ j = 1,\dots,\mu_k.
\end{aligned}
$$

The results in Sections 5, 6, and 7 can be extended to the multiknot case by invoking the appropriate spline interpolation theorems in their multiknot formulations. For example, Theorem 6.2 becomes

THEOREM 8.1. *Let the boundary forms* (6.1) *satisfy Postulate* I. *Then there is a unique monospline* $\tilde{N}$ *satisfying* (8.2) *iff* (6.3) *holds and, if* $n > \sum_{k=1}^{r} \mu_k$, (6.4) *also holds.*

Likewise, the analog of Theorem 6.3 is

THEOREM 8.2. *Consider the class* $\mathscr{C}$ *of quadrature formulas* (8.1) *exact on L-polynomials and with boundary forms* (6.1) *satisfying Postulate* I, (6.8), *and* (6.9). *Then* $\mathscr{C}$ *is nonempty iff* (8.2) *has a solution* $\tilde{N}$. *Moreover,* $\tilde{N}$ *is uniquely determined by* (8.2) *iff*

$$\sum_{k=1}^{r} \mu_k \geq n - (p+q).$$

An important application of Theorem 8.2 is to quadrature formulas of the form

$$Q(u) = \sum_{k=1}^{r} \sum_{j=1}^{m_k} c_{kj} D^{j-1} u(\xi_k);$$

namely, if

$$\sum_{k=1}^{r} \mu_k = n$$

then the best quadrature formula of this type is induced by the monospline $\tilde{M} = L\tilde{N}$, where $\tilde{N}$ is the unique solution to (8.2).

Note added in proof. The author has learned that A. Melkman also obtained Theorem 5.2 by an essentially different means based on Theorem 2.2 in [1]. This approach does not yield the explicit relation between subdeterminants of D_1 and D_{-1} given here.

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